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INSTABILITY OF A LIQUID JET IN A HIGH-FREQUENCY ALTERNATING ELECTRIC FIELD

E. A. Demekhin and S. V. Polyanskikh

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Stability of a liquid (electrolyte) jet in a tangential electric field harmonically oscillating with a high frequency is considered under an assumption of an ideal liquid. It is demonstrated that it is possible to solve the electrodynamic and hydrodynamic parts of the problem inside the jet separately if the Peclet number based on the Debye layer thickness is small. Linear stability of the trivial solution of the problem is studied. A dispersion relation is derived, which is used to study the effect of the amplitude and frequency of electric field oscillations on jet stability. An increase in the amplitude of oscillations is demonstrated to exert a stabilizing effect, whereas an increase in frequency leads to insignificant destabilization of the jet.

Key words: *electrohydrodynamics, electrolyte, linear stability, ideal liquid, microjet.*

Introduction. Capillary jets are known to be unstable and to decompose into individual droplets [1, 2]. A review and detailed derivation of various models of jet behavior can be found in [3].

The problem of the behavior of a capillary jet in an external electric field is a classical problem of electrohydrodynamics and has numerous applications, in particular, as one possible method of liquid spraying (printers, car carburetors, fuel spraying in injectors, etc.). Zeleny [4] performed a pioneering study of liquid jets and droplets under the action of an electric field, and now this publication is considered as a fundamental reference work. At the moment, there are many models that describe various specific features of the processes considered. A situation with an external direct electric field and a charge on the jet surface was studied by many authors (see, e.g., [5–9]) who demonstrated that the presence of a charge on the free surface destabilizes the jet with respect to long-wave disturbances, while the presence of an external tangential field, vice versa, leads to jet stabilization.

The jet behavior depends to a large extent on the type of the working liquid, which can be a dielectric, a conducting liquid, or an electrolyte. Electrolytes are the least studied liquids, despite the fact that they were investigated in experiments [4] and are often used in practice for the purpose of electrospraying.

In some experimental works, in particular, in [10], the authors proposed to use an alternating electric field instead of a direct electric field. Advantages of such an external field are the presence of a new reference parameter (frequency of oscillations), the electric neutrality of liquid droplets being formed, and the absence of undesirable chemical reactions at sufficiently high frequencies of oscillations (above 100 kHz) because the period of oscillations is substantially shorter than the characteristic time of the reaction. The present work is aimed at developing a theory that describes phenomena of this type.

Stability of a jet of an ideal capillary liquid in a tangential electric field oscillating with a frequency of the order of $\tilde{æ}/\tilde{\varepsilon}$ ($\tilde{æ}$ is the electrical conductivity and $\tilde{\varepsilon}$ is the dielectric permeability of the liquid) is considered in the present paper. The motion excited by such a field in the system includes a slowly changing mean component and a fast fluctuating or vibrational component [12]. The resultant motion is a superposition of these components, and the contribution of the vibrational component tends to zero as $\tilde{\omega} \to \infty$ [12].

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Linearizing the system in the neighborhood of the trivial solution, we obtain a problem with eigenvalues depending on the ratio of the dielectric permeabilities of the media, the amplitude and frequency of external field oscillations, the wavenumber and the linear growth rate of disturbances, and the basic dimensionless parameters. The dependence of the stability region on these parameters is studied. It is demonstrated that an increase in the amplitude of external field oscillations leads to jet stabilization owing to constriction of the range of unstable wavelengths and to reduction of the most unstable linear growth rate. In particular, the presence of an oscillating external field always results in jet stabilization. An increase in the frequency of oscillations, vice versa, leads to insignificant destabilization of the jet.

1. Let us consider a jet of an ideal liquid (electrolyte) placed into an external tangential electric field. The processes inside the liquid phase are described by two equations of transportation of negative and positive ions, Poisson's equation for the electric field potential, and hydrodynamic Euler and continuity equations. The liquid is a simple binary electrolyte: $z^+ = -z^- = 1$. The coefficients of diffusion of negative and positive ions are assumed to be identical: $\tilde{D}^+ = \tilde{D}^- = \tilde{D}$. We study the most important case of axisymmetric disturbances of the jet. The $\tilde{x}$ and $\tilde{y}$ axes are directed along the undisturbed axis of the jet and along the jet radius, respectively. In the chosen coordinate system, the full system of equations has the form

$$\frac{\partial \tilde{c}^\pm}{\partial \tilde{t}} + \tilde{u}\,\frac{\partial \tilde{c}^\pm}{\partial \tilde{x}} + \tilde{v}\,\frac{\partial \tilde{c}^\pm}{\partial \tilde{y}} = \tilde{D}\Big\{ \pm \frac{\tilde{F}}{\tilde{R}\tilde{T}}\Big[\frac{\partial}{\partial \tilde{x}}\Big(\tilde{c}^\pm\,\frac{\partial \tilde{\Phi}}{\partial \tilde{x}}\Big) + \frac{1}{\tilde{y}}\,\frac{\partial}{\partial \tilde{y}}\,\Big(\tilde{y}\tilde{c}^\pm\,\frac{\partial \tilde{\Phi}}{\partial \tilde{y}}\Big)\Big] + \frac{\partial^2 \tilde{c}^\pm}{\partial \tilde{x}^2} + \frac{1}{\tilde{y}}\,\frac{\partial}{\partial \tilde{y}}\Big(\tilde{y}\,\frac{\partial \tilde{c}^\pm}{\partial \tilde{y}}\Big)\Big\}; \tag{1}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\,\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\,\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{1}{\tilde{\rho}}\,\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\tilde{F}}{\tilde{\rho}}\,(\tilde{c}^- - \tilde{c}^+)\,\frac{\partial \tilde{\Phi}}{\partial \tilde{x}}; \tag{2}$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\,\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\,\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{1}{\tilde{\rho}}\,\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{\tilde{F}}{\tilde{\rho}}\,(\tilde{c}^- - \tilde{c}^+)\,\frac{\partial \tilde{\Phi}}{\partial \tilde{y}}; \tag{3}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{1}{\tilde{y}}\,\frac{\partial}{\partial \tilde{y}}\,(\tilde{y}\tilde{v}) = 0; \tag{4}$$

$$\frac{\partial^2 \tilde{\Phi}}{\partial \tilde{x}^2} + \frac{1}{\tilde{y}}\,\frac{\partial}{\partial \tilde{y}}\,\Big(\tilde{y}\,\frac{\partial \tilde{\Phi}}{\partial \tilde{y}}\Big) = \frac{\tilde{F}}{\tilde{\varepsilon}}\,(\tilde{c}^- - \tilde{c}^+). \tag{5}$$

Here $\tilde{c}^+$ and $\tilde{c}^-$ are the molar concentrations of cations and anions, respectively, $\tilde{u}$ and $\tilde{v}$ are the axial and radial component of velocity, $\tilde{F}$ is the Faraday number, $\tilde{R}$ is the universal gas constant, $\tilde{T}$ is the temperature, $\tilde{\Phi}$ is the electric field potential, $\tilde{p}$ is the pressure, and $\tilde{\rho}$ is the density of the liquid medium; the tilde indicates the dimensional quantities. The solution is assumed to be electrically neutral far from the interphase boundary:

$$\tilde{c}^\pm = \tilde{c}_\infty. \tag{6}$$

The gas surrounding the jet is assumed to be a dielectric. The electric field potential in the gas phase satisfies the Laplace equation

$$\frac{\partial^2 \bar{\tilde{\Phi}}}{\partial x^2} + \frac{1}{\tilde{y}}\,\frac{\partial}{\partial \tilde{y}}\,\Big(\tilde{y}\,\frac{\partial \bar{\tilde{\Phi}}}{\partial \tilde{y}}\Big) = 0 \tag{7}$$

(hereinafter, the quantities that refer to the exterior problem for the gas are marked by the bar).

To describe the process near the free boundary, it seems reasonable to use the coordinate $\tilde{n}$ along the unit external normal $\boldsymbol{n}$ to the interface and the coordinate $\tilde{\tau}$ along the unit tangential line $\boldsymbol{\tau}$ lying in the plane $(\tilde{x}, \tilde{y})$.

The gas is assumed to be non-conducting; therefore, the flux of negative and positive ions through the free surface is equal to zero:

$$\tilde{n} = 0: \qquad \pm\,\frac{\tilde{c}^\pm \tilde{F}}{\tilde{R}\tilde{T}}\,\frac{\partial \tilde{\Phi}}{\partial \tilde{n}} + \frac{\partial \tilde{c}^\pm}{\partial \tilde{n}} = 0. \tag{8}$$

The potential is continuous across the interface between the phases:

$$\tilde{n} = 0: \qquad \tilde{\Phi} = \bar{\tilde{\Phi}}. \tag{9}$$

Because of the difference in dielectric permeabilities of the media, the derivative of the potential along the normal is discontinuous:

$$\tilde{\varepsilon}\,\frac{\partial \tilde{\Phi}}{\partial \tilde{n}} = \bar{\tilde{\varepsilon}}\,\frac{\partial \bar{\tilde{\Phi}}}{\partial \tilde{n}}. \tag{10}$$

The kinematic condition and the condition of the balance of normal stresses have the form

$$\tilde{v} = \frac{\partial \tilde{h}}{\partial \tilde{t}} + \tilde{u}\,\frac{\partial \tilde{h}}{\partial \tilde{x}}; \tag{11}$$

$$\tilde{p} - \tilde{p}_0 + [\boldsymbol{n}\tilde{T}^E\boldsymbol{n}] = \tilde{\gamma}\tilde{K}, \tag{12}$$

where the square brackets indicate a jump of the quantity on the interface, $\tilde{\gamma}$ is the surface tension, $\tilde{y} = \tilde{h}(\tilde{t}, \tilde{x})$ is the interface equation, $\tilde{K}$ is the mean curvature of the interface, $\tilde{p}_0$ is the pressure in the gas, which can be set to zero without loss of generality, and $\tilde{T}^E$ is the Maxwell–Wagner tensor of electric stresses whose components in an arbitrary orthogonal coordinate system have the form

$$\tilde{T}^E_{ij} = \tilde{\varepsilon}(-(\tilde{E}^2/2)\,\delta_{ij} + \tilde{E}_i\tilde{E}_j).$$

In particular, the normal electric stress (electric pressure) is written in the coordinates $\tilde{n}$ and $\tilde{\tau}$ as

$$\boldsymbol{n}\tilde{T}^E\boldsymbol{n} = \tilde{T}^E_{nn} = (\tilde{\varepsilon}/2)(\tilde{E}_n^2 - \tilde{E}_\tau^2).$$

The problem is closed by the boundary condition at infinity. The external field oscillations are assumed to be harmonic:

$$\tilde{\bar{\Phi}} = \tilde{\Phi}_\infty = -\tilde{x}\tilde{E}_\infty\,\mathrm{e}^{i\tilde{\omega}\tilde{t}} + \text{c. c.} \tag{13}$$

(c. c. indicates a complex conjugate quantity).

Normalizing the lengths, concentrations, electric potential, velocities, time, and pressure to the quantities

$$\tilde{r}_0, \quad \tilde{c}_\infty, \quad \tilde{\Phi}_c = \Big(\frac{\tilde{\gamma}\tilde{r}_0}{\tilde{\varepsilon}}\Big)^{1/2}, \quad \tilde{U}_0 = \Big(\frac{\tilde{\gamma}}{\tilde{\rho}\tilde{r}_0}\Big)^{1/2}, \quad \tilde{t}_0 = \frac{\tilde{r}_0}{\tilde{U}_0} = \Big(\frac{\tilde{\rho}\tilde{r}_0^3}{\tilde{\gamma}}\Big)^{1/2}, \quad \tilde{p}_0 = \tilde{\rho}\tilde{U}_0^2 = \frac{\tilde{\gamma}}{\tilde{r}_0}$$

($\tilde{r}_0$ is the radius of the undisturbed jet), we reduce system (1)–(13) to the dimensionless form as

$$\mathrm{Pe}_c\Big(\frac{\partial c^\pm}{\partial t} + u\,\frac{\partial c^\pm}{\partial x} + v\,\frac{\partial c^\pm}{\partial y}\Big) = \pm\Lambda\Big[\frac{\partial}{\partial x}\Big(c^\pm\,\frac{\partial \Phi}{\partial x}\Big) + \frac{1}{y}\,\frac{\partial}{\partial y}\Big(yc^\pm\,\frac{\partial \Phi}{\partial y}\Big)\Big] + \frac{\partial^2 c^\pm}{\partial x^2} + \frac{1}{y}\,\frac{\partial}{\partial y}\Big(y\,\frac{\partial c^\pm}{\partial y}\Big); \tag{14}$$

$$\begin{gathered}\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + (c^- - c^+)\,\frac{\partial \Phi}{\partial x};\\ \frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + (c^- - c^+)\,\frac{\partial \Phi}{\partial y};\\ \frac{\partial u}{\partial x} + \frac{1}{y}\,\frac{\partial}{\partial y}\,(yv) = 0;\end{gathered} \tag{15}$$

$$\varepsilon_c^2\Big[\frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{y}\,\frac{\partial}{\partial y}\Big(y\,\frac{\partial \Phi}{\partial y}\Big)\Big] = c^- - c^+; \tag{16}$$

$$\frac{\partial^2 \bar{\Phi}}{\partial x^2} + \frac{1}{y}\,\frac{\partial}{\partial y}\Big(y\,\frac{\partial \bar{\Phi}}{\partial y}\Big) = 0; \tag{17}$$

for $n = 0$, we have

$$\pm\Lambda c^\pm\,\frac{\partial \Phi}{\partial n} + \frac{\partial c^\pm}{\partial n} = 0; \tag{18}$$

$$\Phi = \bar{\Phi}; \tag{19}$$

$$\delta\,\frac{\partial \Phi}{\partial n} = \frac{\partial \bar{\Phi}}{\partial n}; \tag{20}$$

$$v = \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x}; \tag{21}$$

$$p - K + [\boldsymbol{n}T^E\boldsymbol{n}] = 0; \tag{22}$$

for $n = \infty$,

$$\bar{\Phi} = -xE_\infty\,\mathrm{e}^{i\omega t} + \text{c. c.} \tag{23}$$

Here, the system parameters

$$\mathrm{Pe}_c = \frac{\tilde{U}_0\tilde{r}_0}{\tilde{D}} = \frac{\tilde{\Phi}_c}{\tilde{D}}\Big(\frac{\tilde{\varepsilon}}{\tilde{\rho}}\Big)^{1/2} = \frac{1}{\tilde{D}}\Big(\frac{\tilde{\gamma}\tilde{r}_0}{\tilde{\rho}}\Big)^{1/2}, \quad \varepsilon_c^2 = \frac{\tilde{\lambda}_{Dc}^2}{\tilde{r}_0^2} = \frac{1}{\tilde{F}\tilde{c}_\infty}\Big(\frac{\tilde{\varepsilon}\tilde{\gamma}}{\tilde{r}_0^3}\Big)^{1/2},$$

$$\Lambda = \frac{\tilde{F}}{\tilde{R}\tilde{T}}\Big(\frac{\tilde{\gamma}\tilde{r}_0}{\tilde{\varepsilon}}\Big)^{1/2}, \quad \delta = \frac{\tilde{\varepsilon}}{\tilde{\bar{\varepsilon}}} \tag{24}$$

are dimensionless quantities. Far from the interface between the phases, we have $c^{\pm} = 1$. It should be noted that there are two characteristic scales for the potential $\tilde{\Phi}$: the electrocapillary potential $\tilde{\Phi}_c$ used above and the electrochemical potential $\tilde{\Phi}_e = \tilde{R}\tilde{T}/\tilde{F}$ (therefore, $\Lambda = \tilde{\Phi}_c/\tilde{\Phi}_e$). If the potential $\tilde{\Phi}$ is normalized to $\tilde{\Phi}_e$ instead of $\tilde{\Phi}_c$, we obtain the following relations instead of the dimensionless parameters (24):

$$\mathrm{Pe}_e = \frac{\tilde{U}_0\tilde{r}_0}{\tilde{D}} = \frac{\tilde{\Phi}_e}{\tilde{D}}\Big(\frac{\tilde{\varepsilon}}{\tilde{\rho}}\Big)^{1/2} = \frac{\tilde{R}\tilde{T}}{\tilde{D}\tilde{F}}\Big(\frac{\tilde{\varepsilon}}{\tilde{\rho}}\Big)^{1/2}, \qquad \varepsilon_e^2 = \frac{\tilde{\lambda}_{De}^2}{\tilde{r}_0^2} = \frac{\tilde{\varepsilon}\tilde{R}\tilde{T}}{\tilde{F}^2\tilde{c}_\infty\tilde{r}_0^2}.$$

The problem is solved under the assumption that $\varepsilon_c = \tilde{\lambda}_{Dc}/\tilde{r}_0 \ll 1$ and $\varepsilon_e = \tilde{\lambda}_{De}/\tilde{r}_0 \ll 1$. In this case, as it follows from Eq. (16), there appears a small parameter at the higher derivative, generating the boundary layer in the vicinity of the free surface of the liquid. Thus, in the limit $\varepsilon_c \to 0$ or $\varepsilon_e \to 0$, the problem decomposes into the external part (far from the interface) and the internal part (near the interface between the phases).

2. Following [11], we consider the solution of system (14)–(23) in a thin Debye layer with a thickness of the order of $O(\varepsilon_e)$. In this zone, the most suitable scale for the potential is the electrochemical potential $\tilde{\Phi}_e$. Using the Debye approximation

$$c^{\pm} = 1 + \hat{c}^{\pm}\,\mathrm{e}^{i\omega t} + \text{c. c.}, \qquad \hat{c}^{\pm} \ll 1, \qquad \Phi = \hat{\Phi}^{\pm}\,\mathrm{e}^{i\omega t} + \text{c. c.},$$

we linearize Eqs. (14), (16), and (18). As a result, in the internal coordinates $\xi = \tau$, $\eta = n/\varepsilon_e$, we obtain

$$i\Lambda\,\mathrm{Pe}_e\,\varepsilon_e^2\omega\hat{c}^{\pm} + \Lambda\,\mathrm{Pe}_e\,\varepsilon_e\Big(\varepsilon_e u\,\frac{\partial\hat{c}^{\pm}}{\partial\xi} + v\,\frac{\partial\hat{c}^{\pm}}{\partial\eta}\Big) = \pm\Big(\varepsilon_e^2\,\frac{\partial^2\hat{\Phi}}{\partial\xi^2} + \frac{\partial^2\hat{\Phi}}{\partial\eta^2}\Big) + \varepsilon_e^2\,\frac{\partial^2\hat{c}^{\pm}}{\partial\xi^2} + \frac{\partial^2\hat{c}^{\pm}}{\partial\eta^2},$$

$$\varepsilon_e^2\,\frac{\partial^2\hat{\Phi}}{\partial\xi^2} + \frac{\partial^2\hat{\Phi}}{\partial\eta^2} = \hat{c}^- - \hat{c}^+,$$

$$\eta = 0: \quad \pm\,\frac{\partial\hat{\Phi}}{\partial\eta} + \frac{\partial\hat{c}^{\pm}}{\partial\eta} = 0, \qquad \eta \to -\infty: \quad \hat{c}^{\pm} = 0.$$

Assuming that $\Lambda\,\mathrm{Pe}_e\,\varepsilon_e \to 0$, we can neglect the convective terms. The assumption $\Omega \equiv \Lambda\,\mathrm{Pe}_e\,\varepsilon_e^2\omega = O(1)$ actually defines the frequency range. Thus, the problem decomposes into two parts in a narrow Debye layer, and the electrodynamic problem can be solved at the beginning. Introducing the bulk density of the charge distribution $\rho = c^+ - c^-$ ($\rho = \hat{\rho}\,\mathrm{e}^{i\omega t} + \text{c. c.}$) and performing a limiting transition $\varepsilon_e \to 0$, we obtain the following system of equations:

$$\frac{d^2\hat{\rho}}{d\eta^2} - (2 + i\Omega)\hat{\rho} = 0, \qquad \frac{d^2\hat{\Phi}}{d\eta^2} = -\hat{\rho},$$

$$\eta = 0: \quad 2\,\frac{d\hat{\Phi}}{d\eta} + \frac{d\hat{\rho}}{d\eta} = 0, \qquad \eta = -\infty: \quad \hat{\rho} = 0.$$

This system yields the relation between the complex amplitudes of the surface charge density $\hat{\sigma}$ and the electric field strength:

$$\hat{\sigma} = -\frac{2}{2 + i\Omega}\,\frac{d\hat{\Phi}}{d\eta}\Big|_{\eta=0}. \tag{25}$$

The expression for the surface charge density (25) is written in the dimensional form as

$$\tilde{\sigma} = -\frac{\tilde{æ}}{\tilde{æ}/\tilde{\varepsilon} + i\tilde{\omega}}\,\frac{\partial\hat{\Phi}}{\partial\tilde{n}}\Big|_{\tilde{n}=0}\,\mathrm{e}^{i\tilde{\omega}\tilde{t}} + \text{c. c.}, \tag{26}$$

where $\tilde{æ} = 2\tilde{F}^2\tilde{D}\tilde{c}_\infty/(\tilde{R}\tilde{T})$ is the electrical conductivity. Under the assumptions made, the value of $\tilde{æ}/\tilde{\varepsilon}$ is of the order of the frequency 10^3–10^7 kHz [13]. For higher values of $\tilde{\omega}$, Eq. (26) acquires the form

$$\tilde{\sigma} = -\frac{i\tilde{æ}}{\tilde{\omega}}\frac{\partial\hat{\Phi}}{\partial\tilde{n}}\Big|_{\tilde{n}=0} \mathrm{e}^{i\tilde{\omega}\tilde{t}} + \text{c. c.} \tag{27}$$

3. In the full formulation (1)–(13), the Euler equations include an electric force with a non-zero mean owing to nonlinearity, though the mean values of the electric potential and concentrations are equal to zero. Thus, the liquid motion can be divided into the mean and oscillatory parts. According to the classical averaging theory [12], the velocity field can be presented in the following form with accuracy to small parameters of the highest order in terms of frequency:

$$\bar{\boldsymbol{U}}(t,x) = \bar{\boldsymbol{U}}_0(t,x) + \bar{\boldsymbol{U}}_1(t,x)\,\mathrm{e}^{2i\omega t} + \text{c. c.}$$

Here $U_1/U_0 \sim 1/\omega$ as $\omega \to \infty$. Therefore, the term $\bar{\boldsymbol{U}}_1$ can be neglected as $\omega \to \infty$. (In what follows, the zero subscript is omitted for convenience.)

Directing ε_c to zero in Eqs. (14)–(23), we obtain an exterior problem. As it follows from Poisson's equation (16) that the solution is electrically neutral ($c^\pm \equiv 1$) in the external region with the liquid depth greater than the Debye length, then the exterior problem decomposes into the electrodynamic and hydrodynamic parts coupled by the boundary conditions on the interface. The external limiting transition $\varepsilon_e \to 0$ in Eq. (16) is equivalent to the assumption that the entire charge is located on the interface and, hence, generates an additional normal stress. With allowance for the expression for the surface charge density (27) obtained in solving the interior problem, condition (20) should be replaced by the expression

$$\delta\Big(1 - \frac{i}{\omega}\Big)\frac{\partial\hat{\Phi}}{\partial n} = \frac{\partial\hat{\bar{\Phi}}}{\partial n}, \tag{28}$$

where $\omega = \tilde{\omega}\tilde{\varepsilon}/\tilde{æ}$ is the dimensionless frequency of oscillations.

The balance of normal stresses is expressed in terms of complex amplitudes. Averaging condition (22) over the fast time ωt and using Eq. (28), we obtain

$$\langle[\boldsymbol{n}T^E\boldsymbol{n}]\rangle = [\langle\boldsymbol{n}T^E\boldsymbol{n}\rangle] = \Big(1 - \frac{1}{\delta}\Big)\Big|\frac{\partial\hat{\Phi}}{\partial\tau}\Big|^2 + \Big(\frac{\delta}{\omega^2} + \delta - 1\Big)\Big|\frac{\partial\hat{\Phi}}{\partial n}\Big|^2,$$

where the averaging is understood as the standard operation

$$\langle\,\cdot\,\rangle = \frac{1}{T}\int_0^T dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt.$$

Condition (22) is finally written as

$$p - K + \Big(1 - \frac{1}{\delta}\Big)\Big|\frac{\partial\hat{\Phi}}{\partial\tau}\Big|^2 + \Big(\frac{\delta}{\omega^2} + \delta - 1\Big)\Big|\frac{\partial\hat{\Phi}}{\partial n}\Big|^2 = 0.$$

In the full formulation, the exterior problem has the following form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}, \qquad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{y}\,v = 0; \tag{29}$$

$$y < h:\quad \frac{\partial^2\hat{\Phi}}{\partial x^2} + \frac{\partial^2\hat{\Phi}}{\partial y^2} + \frac{1}{y}\frac{\partial\hat{\Phi}}{\partial y} = 0, \qquad y > h:\quad \frac{\partial^2\hat{\bar{\Phi}}}{\partial x^2} + \frac{\partial^2\hat{\bar{\Phi}}}{\partial y^2} + \frac{1}{y}\frac{\partial\hat{\bar{\Phi}}}{\partial y} = 0; \tag{30}$$

$$\begin{gathered} y = h:\quad \hat{\Phi} = \hat{\bar{\Phi}}, \qquad \delta\Big(1 - \frac{i}{\omega}\Big)\frac{\partial\hat{\Phi}}{\partial n} = \frac{\partial\hat{\bar{\Phi}}}{\partial n}, \qquad v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}, \\ p + K + \Big(1 - \frac{1}{\delta}\Big)\Big|\frac{\partial\hat{\Phi}}{\partial\tau}\Big|^2 + \Big(\frac{\delta}{\omega^2} + \delta - 1\Big)\Big|\frac{\partial\hat{\Phi}}{\partial n}\Big|^2 = 0; \end{gathered} \tag{31}$$

$$y = \infty:\qquad \hat{\Phi} = -xE_\infty. \tag{32}$$

In this problem, the parameter

$$K = \frac{1}{h(1+h_x^2)^{1/2}} - \frac{h_{xx}}{(1+h_x^2)^{3/2}}$$

is the dimensionless mean curvature of the free surface.

4. Problem (29)–(32) has a trivial solution

$$h = 1, \qquad u = v = 0, \qquad p = 0, \qquad \hat{\Phi} = \hat{\bar{\Phi}} = -E_\infty x.$$

Small perturbations are imposed on the velocity components and on the free boundary:

$$u \sim \hat{u}(y)\,\mathrm{e}^{i\alpha x+\lambda t}, \qquad v \sim \hat{v}(y)\,\mathrm{e}^{i\alpha x+\lambda t}, \qquad p \sim \hat{p}(y)\,\mathrm{e}^{i\alpha x+\lambda t}; \tag{33}$$

$$h \sim 1 + \hat{h}\,\mathrm{e}^{i\alpha x+\lambda t}. \tag{34}$$

By virtue of Eqs. (30)–(32), the potential disturbances have a parametric character in time and are generated by the free boundary perturbation (34). In particular, we obtain the following relation on the interface with accuracy to terms of the highest order of smallness:

$$\Phi\Big|_{y=h} = \Phi\Big|_{y=1} + \frac{\partial \Phi}{\partial y}\Big|_{y=1}\hat{h}\,\mathrm{e}^{i\alpha x+\lambda t} + \ldots\ .$$

Thus, for potential disturbances, we have to assume that

$$\hat{\Phi} \sim -E_\infty x + \varphi(y)\hat{h}\,\mathrm{e}^{i\alpha x+\lambda t}, \qquad \hat{\bar{\Phi}} \sim -E_\infty x + \bar{\varphi}(y)\hat{h}\,\mathrm{e}^{i\alpha x+\lambda t}. \tag{35}$$

Let us now consider the electrostatic part of the problem. Substituting Eqs. (35) into Eqs. (30), we obtain the Bessel equations for the quantities φ and $\bar{\varphi}$. Requiring satisfaction of the potential regularity condition at $y = 0$ and $y = \infty$, we obtain

$$\varphi(y) = C_1 I_0(\alpha y), \qquad \bar{\varphi}(y) = C_2 K_0(\alpha y), \tag{36}$$

where I_0 and K_0 are the zeroth-order Bessel functions. To determine the unknown constants C_1 and C_2, we linearize the first two conditions of Eqs. (31):

$$y = 1\colon \qquad \varphi = \bar{\varphi}, \qquad \delta(1 - i/\omega)(i\alpha E_\infty + \varphi') = i\alpha E_\infty + \bar{\varphi}',$$

whence it follows that

$$C_1 = -\frac{iE_\infty}{I_0}\,\Pi(\alpha,\omega), \quad C_2 = -\frac{iE_\infty}{K_0}\,\Pi(\alpha,\omega), \quad \Pi(\alpha,\omega) = -\frac{\alpha\delta(1 - i/\omega - 1/\delta)}{\delta(1 - i/\omega)I_0'/I_0 - K_0'/K_0}, \tag{37}$$

where the functions I_0 and K_0 are taken at $\xi = \alpha$. Using Eqs. (35)–(37), we obtain the solution of the electrostatic part of the problem of linear stability.

In our further study, we use the expression for the perturbation of the tangential component of the electric field on the interface:

$$\frac{\partial \hat{\Phi}}{\partial \tau} = \frac{\partial \hat{\Phi}}{\partial x}\,n_y - \frac{\partial \hat{\Phi}}{\partial y}\,n_x \sim E_x = -\frac{\partial \hat{\Phi}}{\partial x} \sim E_\infty - i\alpha C_1 I_0(\alpha)\hat{h}\,\mathrm{e}^{i\alpha x+\lambda t}. \tag{38}$$

Substituting Eqs. (33) and (38) into Eqs. (29) and (31) and rejecting terms of higher orders of smallness, we obtain the problem of linear stability

$$\lambda\hat{u} = -i\alpha\hat{p}, \qquad \lambda\hat{v} = -\hat{p}', \qquad i\alpha\hat{u} + \hat{v}' + \hat{v}/y = 0; \tag{39}$$

$$y = 1\colon \qquad \hat{v} = \lambda\hat{h}, \qquad \hat{p} + (1-\alpha^2)\hat{h} + 2E_\infty(1 - 1/\delta)\Pi_R(\alpha,\omega)\hat{h} = 0, \tag{40}$$

where $\Pi_R(\alpha,\omega) = (\Pi(\alpha,\omega) + \Pi^*(\alpha,\omega))/2$. We transform Eqs. (39) to the Bessel equation

$$\hat{u}'' + \hat{u}'/y - \alpha^2\hat{u} = 0. \tag{41}$$

The solution of Eq. (41) satisfying the first condition in Eqs. (40) has the form

$$\hat{u} = i\lambda\,\frac{I_0(\alpha y)}{I_0'(\alpha)}\,\hat{h}, \qquad \hat{v} = \lambda\,\frac{I_0'(\alpha y)}{I_0'(\alpha)}\,\hat{h}, \qquad \hat{p} = -\frac{\lambda^2}{\alpha}\,\frac{I_0(\alpha y)}{I_0'(\alpha)}\,\hat{h}. \tag{42}$$

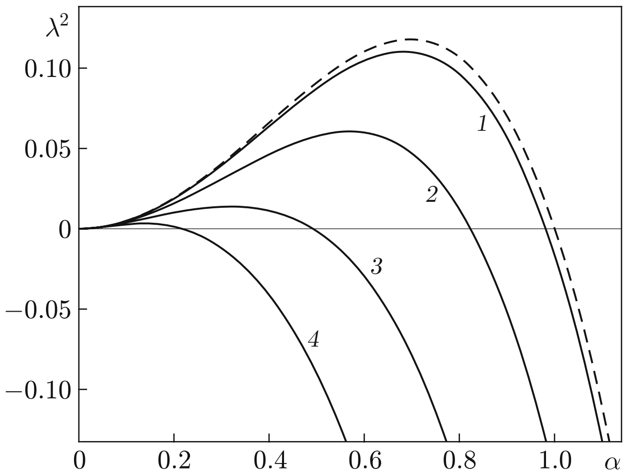

Fig. 1

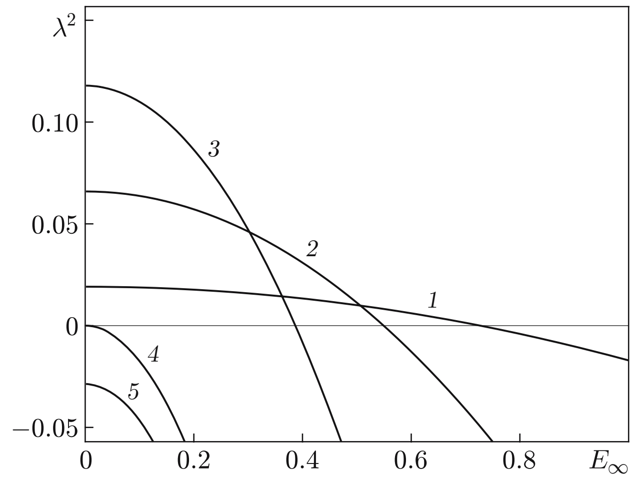

Fig. 2

Fig. 1. Linear growth rate versus the disturbance wavenumber at $\omega = \infty$ and $E_\infty = 0.1$ (1), 0.3 (2), 0.5 (3), and 0.7 (4); the dashed curve shows the results for $E_\infty = 0$.

Fig. 2. Linear growth rate versus the strength of the external electric field at $\omega = \infty$ and $\alpha = 0.20$ (1), 0.40 (2), 0.70 (3), 1.00 (4), and 1.03 (5).

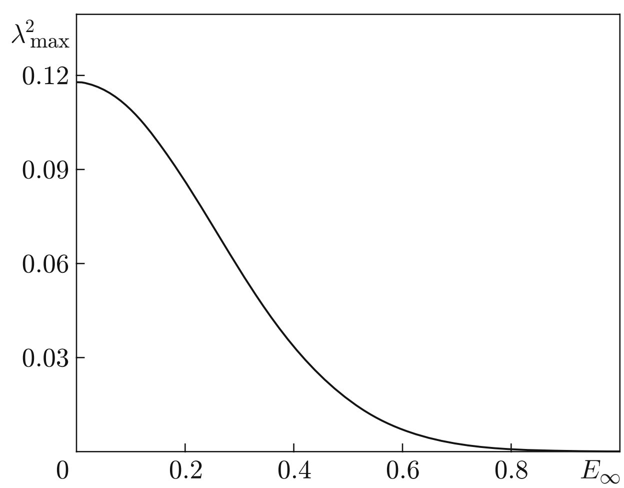

Fig. 3. Linear growth rate corresponding to the most unstable wavenumber of the disturbance versus the external electric field strength.

Substituting the resultant values of pressure (42) into the second boundary condition in Eqs. (40) yields the dispersion relation

$$\lambda^2 = \alpha(1 - \alpha^2) I_0'/I_0 + 2\alpha E_\infty^2 (1 - 1/\delta)(I_0'/I_0)\Pi_R(\alpha, \omega). \tag{43}$$

If there is no electric field applied ($E_\infty = 0$), we obtain the classical Rayleigh result

$$\lambda^2 = \alpha(1 - \alpha^2) I_0'/I_0.$$

As the parameter Π_R is always negative, the second term in the dispersion relation with $E_\infty \neq 0$ yields additional stabilization owing to constriction of the range of unstable wavenumbers. Thus, the presence of an external oscillating field always leads to jet stabilization. As the value of ω is increased, the value of $\Pi_R(\alpha, \omega)$ increases uniformly with respect to α, thus, constricting the stability region, which is still always wider than that at $E_\infty = 0$.

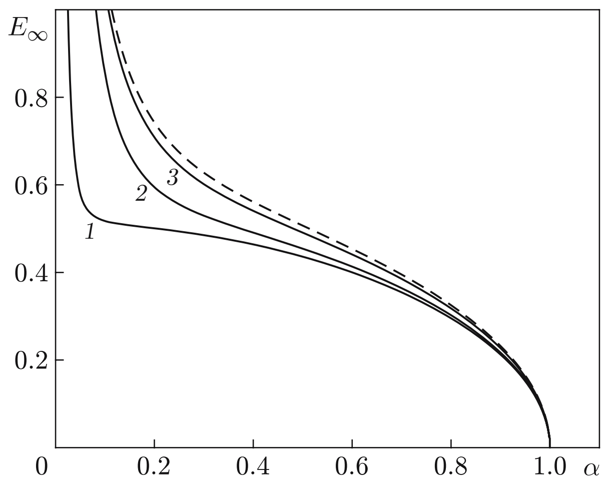

Fig. 4

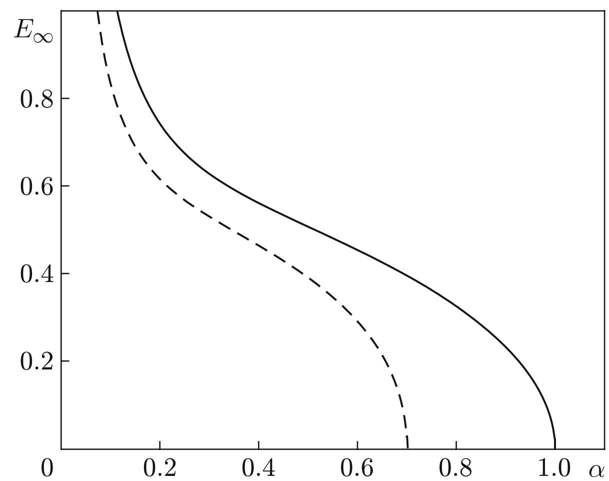

Fig. 5

Fig. 4. Strength of the external electric field versus the disturbance wavenumber for $\omega = 0.1$ (1), 1.0 (2), and 3.0 (3); the dashed curve shows the results for $\omega = \infty$.

Fig. 5. Strength of the external electric field versus the disturbance wavenumber for $\omega = \infty$: the dashed curve is the line of the maximum growth rate.

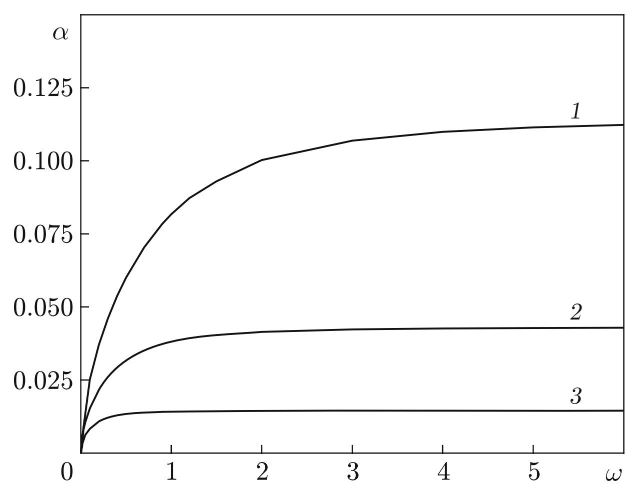

Fig. 6. Neutral stability curves for different values of the external electric field strength: $E_\infty = 1$ (1), 2 (2), and 5 (3).

Figures 1 and 2 show the linear growth rates for different values of parameters. The linear growth rate of the most unstable wavenumber is plotted in Fig. 3 as a function of the external field strength. It is seen that the growth rate tends to zero with increasing electric field strength.

5. The internal parameter of the problem is the disturbance wavenumber α, while the external parameters are the ratio of the dielectric permeabilities δ and the dimensionless frequency of oscillations ω and strength E_∞ of the electric field. All calculations were performed with $\delta = 24$ corresponding to alcohol used in experiments [10].

The expressions for the neutral stability curves separating stable and unstable regions in the plane of parameters are obtained by substituting $\lambda = 0$ into the dispersion relation (43). Thus, the neutral stability curves are described by the relation

$$E_\infty^2 = \frac{\alpha^2 - 1}{2(1 - 1/\delta)} \, \frac{K_0'/K_0 - \delta(1 - i/\omega)I_0'/I_0}{\alpha\delta(1 - i/\omega - 1/\delta)}. \tag{44}$$

Calculations by Eq. (44) show that the presence of a tangential oscillating field stabilizes the jet. For each fixed wavenumber α and an arbitrary frequency of oscillations ω, the jet becomes stabilized when the field strength E_∞ exceeds the value predicted by Eq. (44). An increase in the frequency of field oscillations, vice versa, leads to certain constriction of the stability region and, as a consequence, to jet destabilization owing to expansion of the region of unstable wavenumbers. The results of the corresponding calculations for different values of parameters are plotted in Figs. 4 and 5. An increase in frequency in the range $\omega \gg 1$ leads to only insignificant destabilization of the jet. The neutral stability curves in the plane of parameters (ω, α) for different values of E_∞ are plotted in Fig. 6.

Conclusions. The interest in studying liquid jets placed into an external electric field is caused by their applications in various engineering processes of micro- and nanospraying and in production of ultrathin fibers, as well as by investigations of new types of electrohydrodynamic instabilities of the jets [14, 15]. In particular, the case with an unsteady external electric field is of much importance.

A two-dimensional model constructed in the present work describes the loss of stability of a liquid (electrolyte) jet placed into an oscillating external electric field defined by two reference parameters: amplitude and frequency of oscillations. In the linear approximation, an increase in the amplitude of oscillations is demonstrated to ensure jet stabilization, while an increase in frequency leads to insignificant destabilization of the jet.

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